On the solvability of a Neumann boundary value problem

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1.INTRODUCTION

The purpose of this paper is to establish existence of $C^2[0,1]$-solutions to the scalar Neumann boundary value problem (BVP)

$$\begin{cases} f(t,x,x',x'')=0, & t\in[0,1], \\ x'(0)=a, \quad x'(1)=b, & a\neq b, \end{cases} \tag{N}$$

where the function $f(t,x,p,q)$ and its first derivatives are continuous only on suitable subsets of the set $[0,1]\times R^3$.

The solvability of the homogeneous Neumann problem for the equation $\big(p(t)x'\big)'+f(t,x,x',x'')=y(t)$ has been studied in [5,9,11]. Results, concerning the existence of solutions to the homogeneous and nonhomogeneous Neumann problem for the equation $x''=f(t,x,x',x'')-y(t)$ can be found in [5,10] and [7] respectively. BVPs for the same equation with various linear boundary conditions have been studied in [1,2,7,10]. The results of [12] guarantee the solvability of BVPs for the equation $x''=f(t,x,x',x'')$ with fully linear boundary conditions. BVPs for the equation $f(t,x,x',x'')=0$ with fully nonlinear boundary conditions have been studied in [6]. For results, which guarantee the solvability of the Dirichlet BVP for the same equation, in the scalar and in the vector cases, see [3] and [8] respectively.

Concerning the kind of the nonlinearity of the function $f(t,x,p,q)$, we note that it is assumed semilinear in [1], linear with respect to x, p and q in [2,11] and sublinear in [5], while in [11] f is a Caratheodory function. Finally, in [10] and [12] f is a linear function with respect to q, while with respect to p it is a quadratic function or satisfies Nagumo type growth conditions respectively.

As in [4,6], we use sign conditions to establish a priori bounds for x, x' and x'', where $x(t)\in C^2[0,1]$ is a solution to a suitable family of BVPs containing the problem (N). Using these a priori bounds and applying the topological transversality theorem from [4], we prove our main existence result.

2. BASIC HYPOTHESES

Our results rely on the following three hypotheses.

H1. There are constants $K_x > 0$ and $K_q > 0$ such that

$$f_x(t,x,p,q) \geq K_x \quad \text{for} \quad (t,x,p,q) \in [0,1] \times R \times J_p \times R,$$

$$f_q(t,x,p,q) \leq -K_q \quad \text{for} \quad (t,x,p,q) \in [0,1] \times J_x \times J_p \times R,$$

where $J_x = \left[\min\{0, \frac{a+b}{2}, \frac{a^2}{2(a-b)}\}, \max\{0, \frac{a+b}{2}, \frac{a^2}{2(a-b)}\}\right]$ and $J_p = [\min\{a,b\}, \max\{a,b\}]$.

H2. There are constants $K > 0$, $M > 0$ and a sufficiently small $\varepsilon > 0$ such that

$$f(t,x,p,q) + Kq \geq 0 \text{ for } (t,x,p,q) \in [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times R \times (-\infty, -M),$$

and

$$f(t,x,p,q) + Kq \leq 0 \text{ for } (t,x,p,q) \in [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times R \times (M, \infty),$$

where

$$M_0 = \max\Big\{\frac{e}{e^2-1}(|a-be| + |ae-b|), \frac{Q}{\min\{K, K_q, K_x\}} + \max\{\frac{|a+b|}{2}, \frac{a^2}{2|a-b|}\}\Big\}, \tag{2.1}$$

$Q = \max\Big|\lambda f\big(t,x,p,b-a-(1-\lambda)x\big) - (1-\lambda)K\big(b-a-(1-\lambda)x\big)\Big|$ for $(\lambda,t,x,p) \in [0,1] \times [0,1] \times J_x \times J_p$, and the constants K_x and K_q as well as the sets J_x and J_p are as in **H1**.

H3. $f(t,x,p,q)$ and $f_q(t,x,p,q)$ are continuous and $f_q(t,x,p,q) < 0$ for $(t,x,p,q) \in [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times [-M_1 - \varepsilon, M_1 + \varepsilon] \times [-M_2 - \varepsilon, M_2 + \varepsilon]$, where $M_1 = |a| + M_0 + M$, $M_2 = M_0 + M$, and M_0 and M are as in **H2**.

3. AUXILIARY LEMMAS

In order to obtain our main existence result, we consider the following family of BVPs

$$\begin{cases} K\big(x'' - (1-\lambda)x\big) = \lambda\Big(K\big(x'' - (1-\lambda)x\big) + f\big(t,x,x',(x''-(1-\lambda)x)\big)\Big), \\ x'(0) = a, \quad x'(1) = b, \end{cases} \tag{3.1$_\lambda$}$$

where $\lambda \in [0,1]$, while $K > 0$ is as in **H2**, when **H2** holds, and prove the following two auxiliary lemmas.

LEMMA 3.1. Let **H1** be hold and $x(t) \in C^2[0,1]$ be a solution to $(3.1)_\lambda$, $\lambda \in [0,1]$, where $K > 0$ is an arbitrary constant. Then

$$|x(t)| \leq M_0, \ t \in [0,1],$$

where M_0 is defined by (2.1).

Proof. For $\lambda = 0$, the problem $(3.1)_0$ is of the form

$$x'' - x = 0, \quad x'(0) = a, \quad x'(1) = b.$$

The unique solution to this BVP satisfies the bound

$$|x(t)| \leq \frac{e}{e^2-1}(|a-be| + |ae-b|), \quad t \in [0,1].$$

Let now $\lambda \in (0,1]$. Then the function $y(t) = x(t) - s(t)$, $t \in [0,1]$, where $s(t) = \frac{b-a}{2}t^2 + at$, $t \in [0,1]$, is a solution to the homogeneous boundary value problem

$$K\Big(y''+b-a-(1-\lambda)(y+s)\Big) = \lambda\Big(K\Big(y''+b-a-(1-\lambda)(y+s)\Big)+f\Big(t,y+s,y'+s',y''+b-a-(1-\lambda)(y+s)\Big)\Big),$$

$$y'(0) = y'(1) = 0.$$

From this equation we obtain

$$(1-\lambda)Ky'' = (1-\lambda)^2Ky-(1-\lambda)K(b-a-(1-\lambda)s)+\lambda f\Big(t,y+s,y'+s',y''+b-a-(1-\lambda)(y+s)\Big),$$

$$(1-\lambda)Ky'' = (1-\lambda)^2Ky-(1-\lambda)K(b-a-(1-\lambda)s)+\lambda f\Big(t,y+s,y'+s',y''+b-a-(1-\lambda)(y+s)\Big)-$$
$$-\lambda f\Big(t,s,y'+s',y''+b-a-(1-\lambda)(y+s)\Big)+\lambda f\Big(t,s,y'+s',y''+b-a-(1-\lambda)(y+s)\Big),$$

$$(1-\lambda)Ky'' = (1-\lambda)^2Ky-(1-\lambda)K(b-a-(1-\lambda)s)+\lambda f_x\Big(t,s+\theta_1 y,y'+s',y''+b-a-(1-\lambda)(y+s)\Big)y+$$
$$+\lambda f\Big(t,s,y'+s',y''+b-a-(1-\lambda)(y+s)\Big)-\lambda f\Big(t,s,y'+s',y''+b-a-(1-\lambda)s\Big)+$$
$$+\lambda f\Big(t,s,y'+s',y''+b-a-(1-\lambda)s\Big),$$

$$(1-\lambda)Ky'' = (1-\lambda)^2Ky-(1-\lambda)K(b-a-(1-\lambda)s)+\lambda f_x\Big(t,s+\theta_1 y,y'+s',y''+b-a-(1-\lambda)(y+s)\Big)y+$$
$$-\lambda f_q\Big(t,s,y'+s',y''+b-a-(1-\lambda)s-\theta_2(1-\lambda)y\Big)(1-\lambda)y+$$
$$+\lambda f\Big(t,s,y'+s',y''+b-a-(1-\lambda)s\Big)-\lambda f\Big(t,s,y'+s',b-a-(1-\lambda)s\Big)+\lambda f\Big(t,s,y'+s',b-a-(1-\lambda)s\Big),$$

$$(1-\lambda)Ky'' = (1-\lambda)^2Ky-(1-\lambda)K(b-a-(1-\lambda)s)+\lambda f_x\Big(t,s+\theta_1 y,y'+s',y''+b-a-(1-\lambda)(y+s)\Big)y+$$
$$-\lambda(1-\lambda)f_q\Big(t,s,y'+s',y''+b-a-(1-\lambda)s-\theta_2(1-\lambda)y\Big)y+$$
$$+\lambda f_q\Big(t,s,y'+s',b-a-(1-\lambda)s+\theta_3 y''\Big)y''+\lambda f\Big(t,s,y'+s',b-a-(1-\lambda)s\Big),$$

$$\begin{cases} \Big((1-\lambda)K-\lambda f_q\Big(t,s,y'+s',b-a-(1-\lambda)s+\theta_3 y''\Big)\Big)y'' = \\ \Big((1-\lambda)^2K+\lambda f_x\Big(t,s+\theta_1 y,y'+s',y''+b-a-(1-\lambda)(y+s)\Big)- \\ -\lambda(1-\lambda)f_q\Big(t,s,y'+s',y''+b-a-(1-\lambda)s-\theta_2(1-\lambda)y\Big)\Big)y+ \\ +\lambda f\Big(t,s,y'+s',b-a-(1-\lambda)s\Big)-(1-\lambda)K(b-a-(1-\lambda)s), \end{cases} \tag{3.2}$$

where $0 < \theta_i < 1$, $i = 1,2,3$.

Next, suppose that $|y(t)|$ achieves its maximum at $t_0 \in (0,1)$. Then the function $z = y^2(t)$ has also a maximum at t_0. Consequently, we see that

$$0 \geq z''(t_0) = 2y(t_0)y''(t_0). \tag{3.3}$$

Using the fact that $y'(t_0) = 0$, from (3.2) we obtain

$$\begin{cases} \Big((1-\lambda)K - \lambda f_q\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0 + \theta_3 y_0''\big)\Big) y_0'' = \\ \Big((1-\lambda)\big((1-\lambda)K - \lambda f_q\big(t_0, s_0, s_0', y_0'' + b - a - (1-\lambda)s_0 - \theta_2(1-\lambda)y_0\big)\big) + \\ \lambda f_x\big(t_0, s_0 + \theta_1 y_0, s_0', y_0'' + b - a - (1-\lambda)(y_0 + s_0)\big)\Big) y_0 + \\ +\lambda f\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0\big) - (1-\lambda)K(b-a-(1-\lambda)s_0), \end{cases} \tag{3.4}$$

where $s_0 = s(t_0)$, $s_0' = s'(t_0)$, $y_0 = y(t_0)$, $y_0'' = y''(t_0)$.

On the other hand, in view of **H1**, we have

$$\begin{cases} (1-\lambda)\big((1-\lambda)K - \lambda \overline{f}_q\big) + \lambda \overline{f}_x \geq \min\{(1-\lambda)K - \lambda \overline{f}_q, \overline{f}_x\} \geq \\ \qquad \min\{K, -\overline{f}_q, \overline{f}_x\} \geq \min\{K, K_q, K_x\}, \end{cases} \tag{3.5}$$

where

$$\overline{f}_q = f_q\big(t_0, s_0, s_0', y_0'' + b - a - (1-\lambda)s_0 - \theta_2(1-\lambda)y_0\big),$$
$$\overline{f}_x = f_x\big(t_0, s_0 + \theta_1 y_0, s_0', y_0'' + b - a - (1-\lambda)(y_0 + s_0)\big).$$

Suppose now that $|y(t_0)| > \frac{Q}{\min\{K, K_x, K_q\}}$. Then, from (3.4) and (3.5) it follows that

$$\begin{cases} \Big((1-\lambda)K - \lambda f_q\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0 + \theta_3 y_0''\big)\Big) y_0'' \geq \min\{K, K_q, K_x\} y(t_0) + \\ +\lambda f\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0\big) - (1-\lambda)K(b-a-(1-\lambda)s_0) \end{cases} \tag{3.6}$$

if $y(t_0) > \frac{Q}{\min\{K, K_x, K_q\}}$ and

$$\begin{cases} \Big((1-\lambda)K - \lambda f_q\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0 + \theta_3 y_0''\big)\Big) y_0'' \leq \min\{K, K_q, K_x\} y(t_0) + \\ +\lambda f\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0\big) - (1-\lambda)K(b-a-(1-\lambda)s_0) \end{cases} \tag{3.7}$$

if $y(t_0) < -\frac{Q}{\min\{K, K_x, K_q\}}$. Multiplying (3.6) and (3.7) by $y(t_0)$, we obtain

$$\Big((1-\lambda)K - \lambda f_q\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0 + \theta_3 y_0''\big)\Big) y_0'' y_0 \geq y_0\big(\min\{K, K_q, K_x\} y_0 - Q\big) > 0,$$

$$\Big((1-\lambda)K - \lambda f_q\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0 + \theta_3 y_0''\big)\Big) y_0'' y_0 \geq y_0\big(\min\{K, K_q, K_x\} y_0 + Q\big) > 0.$$

respectively. Finally, since $f_q\big(t_0, s_0, s_0', b-a-(1-\lambda)s_0 + \theta_3 y_0''\big) < 0$, we conclude that

$$y_0'' y_0 > 0,$$

which contradicts (3.3). Thus, we infer that if $|y(t)|$ achieves its maximum in $(0,1)$, then

$$|y(t)| \leq \frac{Q}{\min\{K, K_x, K_q\}} \quad \text{for} \quad t \in [0,1] \quad \text{and} \quad \lambda \in (0,1].$$

Let $|y(1)|$ be the maximum of $|y(t)|$ and suppose that $|y(1)| > \frac{Q}{\min\{K, K_x, K_q\}}$. Following the above reasoning and the fact that $y'(1) = 0$, we obtain

$$y(1)y''(1) > 0.$$

If $y(1) > 0$, then $y''(1) > 0$ and so $y'(t)$ must be a strictly increasing function for $t \in U_1$, where $U_1 \subset [0,1]$ is a sufficiently small neighbourhood of $t = 1$. So, we see that

$$y'(t) < y'(1) = 0 \quad \text{for} \quad t \in U_1 \setminus \{1\},$$

i.e. $y(t)$ is a strictly decreasing function for $t \in U_1$. Therefore, $y(1) = |y(1)|$ can not be the maximum of $|y(t)|$ on $[0,1]$, which is a contradiction. Assume next that $y(1) < 0$. Then a similar to the above arguments lead again to a contradiction. Thus, we see that

$$|y(1)| \leq \frac{Q}{\min\{K, K_x, K_q\}}.$$

The inequality

$$|y(0)| \leq \frac{Q}{\min\{K, K_x, K_q\}}$$

can be obtained in the same manner. Consequently, the solutions of $(3.1)_\lambda$, $\lambda \in (0,1]$, satisfy the bound

$$|x(t)| \leq \frac{Q}{\min\{K, K_x, K_q\}} + \max\Big\{\frac{a^2}{2|a-b|}, \frac{|a+b|}{2}\Big\}, \quad t \in [0,1],$$

and the proof of the lemma is complete.□

LEMMA 3.2. Let **H1** and **H2** be hold and let $x(t) \in C^2[0,1]$ be a solution to $(3.1)_\lambda$, $\lambda \in [0,1]$, where K is as in **H2**. Then:

(a)

$$|x''(t) - (1-\lambda)x(t)| \leq M, \quad |x''(t)| \leq M_2, \quad t \in [0,1],$$

where $M_2 = M_0 + M$;

(b)

$$|x'(t)| \leq M_1, \; t \in [0,1],$$

where $M_1 = |a| + M_0 + M$.

Proof. (a) Suppose that there exists a $(t_0, \lambda_0) \in [0,1] \times [0,1]$ or a $(t_1, \lambda_1) \in [0,1] \times [0,1]$ such that

$$x''(t_0) - (1-\lambda_0)x(t_0) < -M \quad \text{or} \quad x''(t_1) - (1-\lambda_1)x(t_1) > M.$$

By Lemma 3.1, we have

$$|x(t)| \leq M_0 \;\text{ for }\; t \in [0,1]. \tag{3.8}$$

In particular, (3.8) holds for t_0 or t_1. Thus, in view of **H2**, we have

$$0 > K\big(x''(t_0) - (1-\lambda_0)x(t_0)\big) = \lambda_0\Big(K\big(x''(t_0) - (1-\lambda_0)x(t_0)\big) +$$

$$+ f\big(t_0, x(t_0), x'(t_0), x''(t_0) - (1-\lambda_0)x(t_0)\big)\Big) \geq 0$$

or

$$0 < K\big(x''(t_1) - (1-\lambda_1)x(t_1)\big) = \lambda_1\Big(K\big(x''(t_1) - (1-\lambda_1)x(t_1)\big) +$$

$$+ f\big(t_1, x(t_1), x'(t_1), x''(t_1) - (1-\lambda_1)x(t_1)\big)\Big) \leq 0,$$

respectively, which is a contradiction. The obtained contradiction shows that

$$-M \leq x''(t) - (1-\lambda)x(t) \leq M \;\text{ for }\; t \in [0,1] \text{ and } \lambda \in [0,1],$$

and therefore

$$-(M_0 + M) \leq x''(t) \leq M_0 + M \quad \text{for} \quad t \in [0,1],$$

which proves (a).

(b) Observe that, by the mean value theorem, for each $t \in (0,1]$ there is a $\xi \in (0,t)$ such that

$$x'(t) - x'(0) = x''(\xi)t.$$

Since, in view of (a), we have $|x''(\xi)| \leq M_0 + M$, from the last formula we find that

$$|x'(t)| \leq |x'(0)| + |x''(\xi)| \leq |a| + M_0 + M, \ t \in [0,1],$$

which proves (b) and completes the proof of the lemma.□

4. THE MAIN RESULT

Our main result is the following existence theorem, the proof of which is based on the lemmas of the previous section and the topological transversality theorem from [4].

THEOREM 4.1. Let **H1**, **H2** and **H3** be hold. Then the problem (N) has at least one solution in $C^2[0,1]$.

Proof. For any $(\lambda, t, x, p, q) \in [0,1] \times [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times [-M_1 - \varepsilon, M_1 + \varepsilon] \times [-M_2 - \varepsilon, M_2 + \varepsilon]$ consider the function $h(\lambda, t, x, p, q) = \lambda\big(Kq + f(t, x, p, q)\big) - Kq$, where M_i, $i = 0, 1, 2$ are the constants for which, in view of Lemmas 3.1 and 3.2, each $C^2[0,1]$-solution $x(t)$ to $(3.1)_\lambda$, $\lambda \in [0,1]$, satisfies the bounds

$$|x(t)| \leq M_0, \quad |x'(t)| \leq M_1, |x''(t) - (1-\lambda)x(t)| \leq M, \quad \text{and} \quad |x''(t)| \leq M_2, \ \text{for} \ t \in [0,1], \tag{3.9}$$

respectively. Since $M_2 > M$, in view of **H2**, we obtain

$$h(\lambda, t, x, p, -M_2 - \varepsilon) > 0 \quad \text{and} \quad h(\lambda, t, x, p, M_2 + \varepsilon) < 0$$

for $(\lambda, t, x, p) \in [0,1] \times [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times [-M_1 - \varepsilon, M_1 + \varepsilon]$. Besides, by **H3,** we see that $h(\lambda, t, x, p, q)$ and $h_q(\lambda, t, x, p, q)$ are continuous functions and $h_q(\lambda, t, x, p, q) < 0$ for $(\lambda, t, x, p, q) \in [0,1] \times [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times [-M_1 - \varepsilon, M_1 + \varepsilon] \times [-M_2 - \varepsilon, M_2 + \varepsilon]$. Therefore, there is a unique function $G(\lambda, t, x, p)$, which is continuous on the set $[0,1] \times [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times [-M_1 - \varepsilon, M_1 + \varepsilon]$ and such that

$$q = G(\lambda, t, x, p), \ \ (\lambda, t, x, p) \in [0,1] \times [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times [-M_1 - \varepsilon, M_1 + \varepsilon],$$

is equivalent to the equation

$$h(\lambda, t, x, p, q) = 0, \ (\lambda, t, x, p, q) \in [0,1] \times [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times [-M_1 - \varepsilon, M_1 + \varepsilon] \times [-M_2 - \varepsilon, M_2 + \varepsilon].$$

So, since $|x''(t) - (1-\lambda)x(t)| \leq M < M_2 + \varepsilon$ for $t \in [0,1]$ and $\lambda \in [0,1]$, the family $(3.1)_\lambda$ is equivalent to the following families of BVPs

$$\begin{cases} x'' - (1-\lambda)x = G(\lambda, t, x, x'), & t \in [0,1], \\ x'(0) = a, \ x'(1) = b, \end{cases} \tag{3.10$_\lambda$}$$

and

$$\begin{cases} x'' - (2-\lambda)x = G(\lambda, t, x, x') - x, & t \in [0,1], \\ x'(0) = a, \ x'(1) = b, \end{cases} \tag{3.11$_\lambda$}$$

$\lambda \in [0,1]$. Note that from $h(0,t,x,p,0) = 0$ it follows that

$$G(0,t,x,p) = 0 \text{ for } (t,x,p) \in [0,1] \times [-M_0 - \varepsilon, M_0 + \varepsilon] \times [-M_1 - \varepsilon, M_1 + \varepsilon]. \tag{3.12}$$

Now, for $C_B^2[0,1] = \{x(t) \in C^2[0,1] : x'(0) = a,\ x'(1) = b\}$ define the set

$$U = \Big\{x \in C_B^2[0,1] : |x| < M_0 + \varepsilon,\ |x'| < M_1 + \varepsilon,\ |x''| < M_2 + \varepsilon\Big\}$$

and then for $\lambda \in [0,1]$ define the maps

$$G_\lambda : C^1[0,1] \to C[0,1] \quad \text{by} \quad (G_\lambda x)(t) = G\Big(\lambda, t, x(t), x'(t)\Big) - x(t),\ t \in [0,1],$$

$$j : C_B^2[0,1] \to C^1[0,1] \quad \text{by} \quad jx = x \quad \text{and} \quad L_\lambda : C_B^2[0,1] \to C[0,1] \quad \text{by} \quad L_\lambda x = x'' - (2-\lambda)x.$$

Since L_λ, $\lambda \in [0,1]$, is a continuous, linear, one-to-one map of $C_B^2[0,1]$ onto $C[0,1]$, the map L_λ^{-1}, $\lambda \in [0,1]$, exists and is continuous. In addition, G_λ, $\lambda \in [0,1]$, is a continuous and j is a completely continuous embedding. Since $j(\overline{U})$ is a compact subset of $C^1[0,1]$, and G_λ, $\lambda \in [0,1]$, and L_λ^{-1}, $\lambda \in [0,1]$, are continuous on $j(\overline{U})$ and $G_\lambda(j(\overline{U}))$ respectively, the homotopy

$$H : \overline{U} \times [0,1] \to C^2[0,1] \text{ defined by } H(x,\lambda) \equiv H_\lambda(x) \equiv L_\lambda^{-1} G_\lambda j(x)$$

is compact. Besides, the equation

$$L_\lambda^{-1} G_\lambda j(x) = x \quad \text{for } x \in \overline{U} \quad \text{yields} \quad L_\lambda x = G_\lambda j x,$$

coincides with the BVP $(3.11)_\lambda$. Thus, the fixed points of $H_\lambda(x)$ are solutions to $(3.11)_\lambda$. But, by (3.9), the solutions to $(3.11)_\lambda$ are elements of U. Consequently, $H_\lambda(x), \lambda \in [0,1]$, is a fixed point free on ∂U, i.e. $H_\lambda(x)$ is an admissible map for all $\lambda \in [0,1]$. Finally, using (3.12), we see that the map H_0 is a constant map, i.e. $H_0(x) \equiv l$, where l is the unique solution to the BVP

$$x'' - 2x = -x, \quad x'(0) = a, \quad x'(1) = b.$$

From the fact that $l \in U$ it follows that H_0 is an essential map (see, [4]). By the topological transversality theorem (see, [4]), $H_1 = L_1^{-1} G_1 j$ is also essential. So, the problem $(3.11)_1$ has a $C^2[0,1]$-solution. That is, $(3.10)_1$ has a $C^2[0,1]$-solution. To complete the proof, remark that the problem $(3.10)_1$ is equivalent to $(3.1)_1$, which coincides with the problem (N).□

We conclude with the following example, which illustrates our main result.

EXAMPLE 4.1. Consider the boundary value problem

$$1 - (1.5 - t)x'' - tx''^5 - \cos x' + x = 0,$$

$$x'(0) = 0, \quad x'(1) = 10^{-4}.$$

Clearly, **H1** holds for $K_x = 1$, $K_q = 0.5$, $J_x = [0, 5.10^{-5}]$ and $J_p = [0, 10^{-4}]$. Next, observe that

$$5.10^{-5} \le 10^{-4} - (1-\lambda)x \le 10^{-4} \quad \text{for} \quad x \in J_x$$

and choose $K = 0.5$. Then, from

$$-1,5.10^{-4} - 10^{-20} \le -(1,5 - t)\Big(10^{-4} - (1-\lambda)x\Big) - t\Big(10^{-4} - (1-\lambda)x\Big)^5 \le -2,5.10^{-5}$$

for $(\lambda, t, x) \in [0,1] \times [0,1] \times J_x$ and

$$0 \le 1 - \cos p \le 5.10^{-9} \text{ for } p \in J_p$$

it follows that

$$-16.10^{-5} \leq 1-(1,5-t)\big(10^{-4}-(1-\lambda)x\big)-t\big(10^{-4}-(1-\lambda)x\big)^5-\cos p+x \leq 25.10^{-6}+5.10^{-9}$$

for $(\lambda,t,x,p) \in [0,1]\times[0,1]\times J_x \times J_p$. Therefore $Q=\max\{16.10^{-5},\, 0,5.10^{-4}\}=16.10^{-5}$. Note that

$$M_0=\max\Big\{\frac{e}{e^2-1}(|10^{-4}e|+|10^{-4}|),\frac{16.10^{-5}}{\min\{1,\frac{1}{2}\}}+5.10^{-5}\Big\}=37.10^{-5}$$

and, as it is easy to see, **H2** and **H3** hold for $M=5$ and $\varepsilon=3.10^{-5}$. Thus, we can apply Theorem 4.1 to conclude that the considered problem has a solution in $C^2[0,1]$.

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